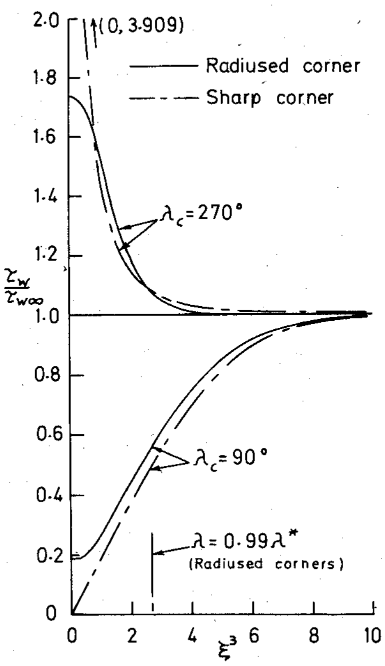

Fig. 4 Wall shear stress.

Equations (8) are slightly different in appearance from the corresponding results in Ref. 4 for the sharp corner, but it is easy to show that they represent exactly the same physical situation. In other words, the physical boundary conditions are dependent only on the angle λ^* and not on whether the corner is sharp or radiused.

Note that the equations determining the flow in a sharp streamwise corner are obtained directly from the above analysis by setting $\lambda = \lambda^*$ for all ξ^3 and replacing the conditions in Eq. (8) for $\xi^3 = 0$ with the more general form

$$u_{,3} - \sin\lambda u_{,2} = 0, \quad \psi_{,3} - \sin\lambda \psi_{,2} + \sin\lambda \phi_{,3} = 0$$

$$\phi = 0, \quad A = 0$$

Choosing $\lambda = \lambda^* \tanh\xi^3$, the methods of Ref. 4 were used to solve Eqs. (2), (3), (6), (7), and (8) numerically for corners of included angles 90, 135, 225, and 270 deg (i.e., $\lambda^* = 45$, 22.5, -22.5, and -45 deg).

The most significant results are the streamwise and Cartesian cross-flow velocity components u and $v(2)$ at the symmetry plane $y^3 = \xi^3 = 0$ and the shear stress $\tau_w(\xi^3)$ on the wall $\xi^2 = 0$, where

$$\tau_w = \tau_{w\infty} \frac{\cos\lambda^*}{\cos\lambda} \frac{(\partial u/\partial \xi^2)_{\xi^2=0}}{f''(0)}$$

and

$$\tau_{w\infty} = \tau_w(\infty) = f''(0)/\cos\lambda^*$$

These quantities are shown in Figs. 2-4 for 90- and 270-deg corners. Included for comparison are the corresponding sharp corner results also obtained from the present program. These agree exactly with the results of Ref. 4 and therefore, as demonstrated in that paper, they are also in very good agreement with the 90-deg sharp corner results given in Ref. 3.

References

[1] Rubin, S. G., "Incompressible Flow Along a Corner," *Journal of Fluid Mechanics*, Vol. 26, Pt. 1, 1966, pp. 97-110.

[2] Desai, S. S. and Mangler, K. W., "Incompressible Laminar Boundary Layer Along a Corner Formed by Two Intersecting Planes," RAE Tech. Rept. 74062, 1974.

[3] Ghia, K. N., "Incompressible Streamwise Flow Along an Unbounded Corner," *AIAA Journal*, Vol. 13, July 1975, pp. 902-907.

[4] Barclay, W. H. and Ridha, A. H., "Flow in Streamwise Corners of Arbitrary Angle," *AIAA Journal*, Vol. 18, Dec. 1980, pp. 1413-1420.

AIAA 82-4121

Simple Relations for the Stability of Heated-Water Laminar Boundary Layers

J. Aroesty,* W. S. King,† G. M. Harpole,‡
W. W. Matyskiela,§ C. Gazley,‡ and A. R. Wazzan¶
The Rand Corporation, Santa Monica, Calif.

I. Introduction

RECENT theoretical and experimental studies suggest that surface heating in water holds promise for enhancing boundary-layer stability by retarding the growth of Tollmien-Schlichting instabilities, or by increasing the surface area over which two-dimensional, infinitesimal disturbances are damped. In the absence of a comprehensive theory of boundary-layer transition, linear stability theory currently provides the sole analytic guide for manipulating mean flow velocity profiles to delay transition.

Even for preliminary engineering applications, no single parameter describes the stability of a particular velocity profile. However, the minimum critical Reynolds number can be extremely useful, both as a precise measure of the extent over which all two-dimensional, infinitesimal disturbances are damped and as a simple qualitative surrogate for the stability characteristics of a particular velocity profile.[1]

Lin[2] developed a simple set of asymptotic relations for determining the minimum critical Reynolds number from the velocity profile of a constant property, laminar boundary layer. The Lin relations were widely used[3] until the advent of computer-based schemes for the solution of the Orr-Sommerfeld equation. In 1946, Lees and Lin[4] extended the original Lin relations to the compressible flow of air, and later, Dunn and Lin[5] and Mack[6] presented further compressible extensions of the original Lin analysis for the neutral curve. However, special difficulties associated with compressibility diminished the role of this analytic approach in high-speed gas dynamics, and numerical computation preempted analytic solution for the airflows.

For heated-water boundary layers, in practice, the situation is simpler than for compressible flows. Water density is nearly constant, temperature and viscosity fluctuations have little effect on stability, and the primary departure from the constant property incompressible flow originally considered by Lin is the variation of mean flow viscosity with temperature.[7,8]

In this Note, we show how the Lin relations and Dunn-Lin theory may be used, after slight modification, to estimate the minimum critical Reynolds number for heated boundary layers. The results of our modified Dunn-Lin analysis are then compared with values obtained from numerical solutions of the Orr-Sommerfeld equation for heated-water boundary layers. Once the accuracy of these new relations is established, they are then used to indicate the influence of changing temperature levels on the minimum critical Reynolds number.

Received April 27, 1981; revision received Oct. 23, 1981.

*Senior Staff, Engineering and Applied Sciences Department.

†Senior Staff, Engineering and Applied Sciences Department. Associate Fellow AIAA.

‡Consultant, Engineering and Applied Sciences Department.

§Staff, Engineering and Applied Sciences Department.

¶Consultant, Engineering and Applied Sciences Department; also Professor, Department of Chemical, Nuclear and Thermal Engineering, The University of California at Los Angeles, Los Angeles, Calif.

The surprisingly powerful effect of axial surface temperature gradient in maintaining stability is also illustrated for the case of unfavorable pressure gradient. Because of their accuracy and simplicity, the new relations may be useful in preliminary feasibility and optimization studies of laminar flow hydrodynamics.

II. Method

In the Dunn-Lin approach, λ is a small quantity reflecting the curvature of the velocity profile between wall and critical layer. This parameter is defined to include both variable kinematic viscosity as well as profile curvature. We adopt this definition of λ, neglecting density variation, or

$$\lambda \equiv \frac{1}{c}\frac{dU}{d\eta}(0)\left(\frac{c}{\mu(0)}\right)^{-1/2}\left[\frac{3}{2}\int_0^{\eta_c}\left(\frac{c-U}{\mu}\right)^{-1/2}d\eta\right]-1 \tag{1}$$

where μ is viscosity nondimensionalized by its freestream value, and η is y nondimensionalized by boundary-layer thickness. This definition follows from the application of the modified Tollmien variable,

$$\zeta \equiv (\alpha R)^{1/3}\left[\int_{\eta_c}^{\eta}\frac{3}{2}\left(\frac{U-c}{\mu}\right)^{1/2}d\eta\right]^{2/3} \tag{2}$$

to obtain the dominant viscous terms in the asymptotic solution of the Orr-Sommerfeld equation. Lin found that the minimum critical Reynolds number is approximated when the wall ($\eta=0$) value of ζ is given by -3.21,

$$R_c=(3.21)^3\left(\frac{dU}{d\eta}(0)\right)^2\frac{\mu(0)}{\alpha c^3(1+\lambda)^2} \tag{3}$$

Using available constant property numerical solutions, Lin estimated α as $\alpha \cong (dU/d\eta)(0)c$, neglected terms of $0(\lambda)$ in Eq. (3), and adjusted constants to develop the relation $R_c=25(\partial U/\partial y)(0)c^{-4}$. We proceed in the same spirit, but choose constants in accord with accurate recent computation for the Blasius profile. The updated relation then becomes

$$R_c \cong 28\frac{dU}{d\eta}(0)\mu(0)c^{-4} \tag{4}$$

Lin's relations for $\nu(c)$ and y_c are unchanged:

$$\nu(c)(1-2\lambda)=0.58 \tag{5}$$

and

$$\nu(c)=-\pi\frac{\partial U}{\partial y}(0)U(y_c)\frac{\partial^2 U}{\partial y^2}(y_c)\left(\frac{\partial U}{\partial y}(y_c)\right)^{-3} \tag{6}$$

Because λ is small, it can be approximated with sufficient accuracy by

$$\lambda=\frac{2}{5}\left\{\left[1-\frac{(dU/d\eta)(\eta_c)}{(dU/d\eta)(0)}\right]+\frac{1}{2}\left(1-\frac{\mu(\eta_c)}{\mu(0)}\right)\right\} \tag{7}$$

This follows from assuming a linear viscosity profile, a parabolic velocity profile, and then evaluating the dominant term in Eq. (2). The required modifications of the Lin relations are Eqs. (5-7) for determining y_c and c, and Eq. (4) for determining the minimum critical Reynolds number. Because of sensitivity to profile curvature, accurate laminar velocity profiles in the wall region are required.

III. Results

A comparison between values of R_c^* (where R^* is the Reynolds number based on displacement thickness) calculated by the present approximate method and those calculated by numerical computation of the Orr-Sommerfeld equation for exact boundary-layer profiles is shown in Fig. 1. The curves are based on published results of Wazzan and co-workers[7,10] for Falkner-Skan similarity flows and various surface overheats. The symbols correspond to the current results. Although no single profile shape factor can uniquely correlate R_c^* when both pressure gradient and surface heating are present, $H \equiv \delta^*/\theta$ remains a convenient parameter for presenting these results. (δ^* is displacement thickness and θ is momentum thickness.)

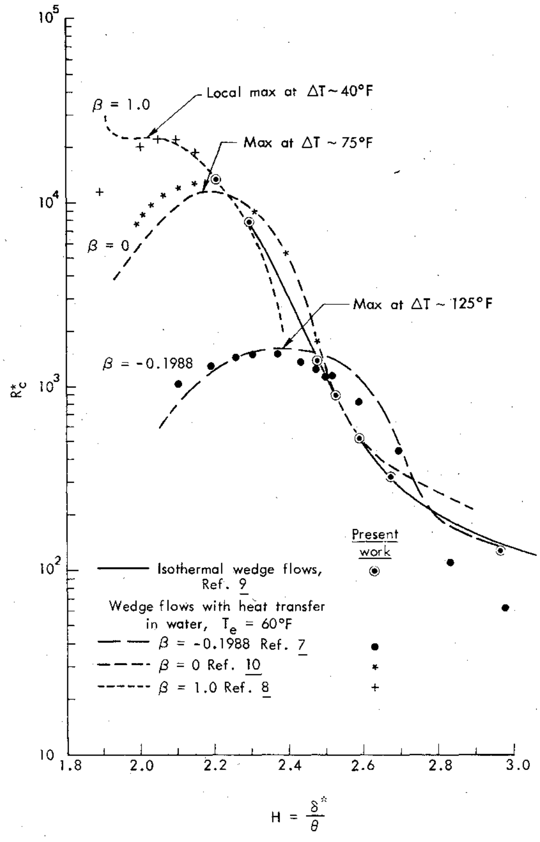

Fig. 1 Comparison between modified Dunn-Lin numerical solutions of the Orr-Sommerfeld equation.

The open circles are computed by our approximate formulas for Falkner-Skan flows at zero ΔT. The agreement between exact calculation and the proposed method is remarkable in the "practical" range of H between 2.2 and 3. The asterisks correspond to the case of $\beta=0$ (flat plate) and various surface overheating, where β is the Falkner-Skan pressure gradient parameter. Note that the agreement between the exact and approximate methods is again satisfactory in the region between $H=2.59$ and $H\approx 2.2$, where a stability reversal occurs as ΔT increases past 24°C (75°F). The method agrees with exact calculations in predicting the existence and location of the maximum attainable critical Reynolds number, both for this case and for the case of $\beta=-0.1988$ (closed circles). For $\beta=0$, the method appears to overpredict R_c^* in the region of large surface overheating corresponding to values of H less than 2.3. The case of $\beta=1.0$ (crosses), corresponding to an extremely favorable pressure gradient, is more problematic. Exact calculations indicate that R_c^* exhibits a local maximum of 2.2×10^4 at $H=2.05$ and then increases again with further surface overheat. The approximate method, however, exhibits an absolute maximum at $H=2.05$, corresponding to $\Delta T=4.44$°C (40°F). There is a good agreement between the exact and approximate values up to this value of surface overheat.

Figure 2 shows the variation of Rx_c (where Rx is the Reynolds number based on external conditions and axial

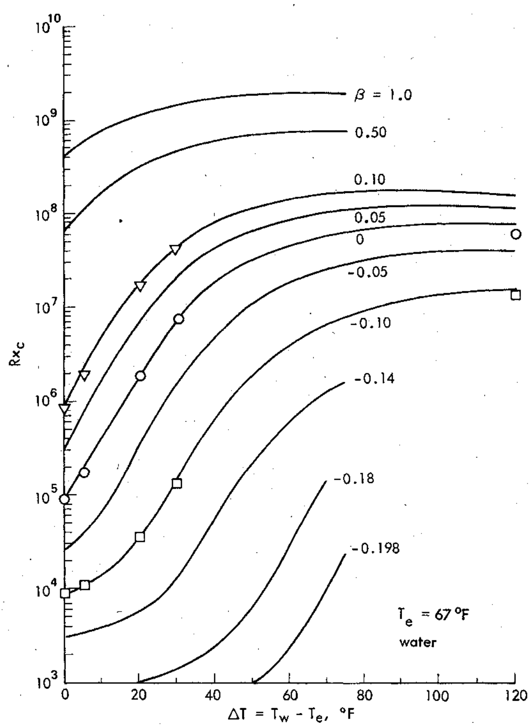

Fig. 2 The minimum critical Reynolds number computed by the modified Dunn-Lin approximation.

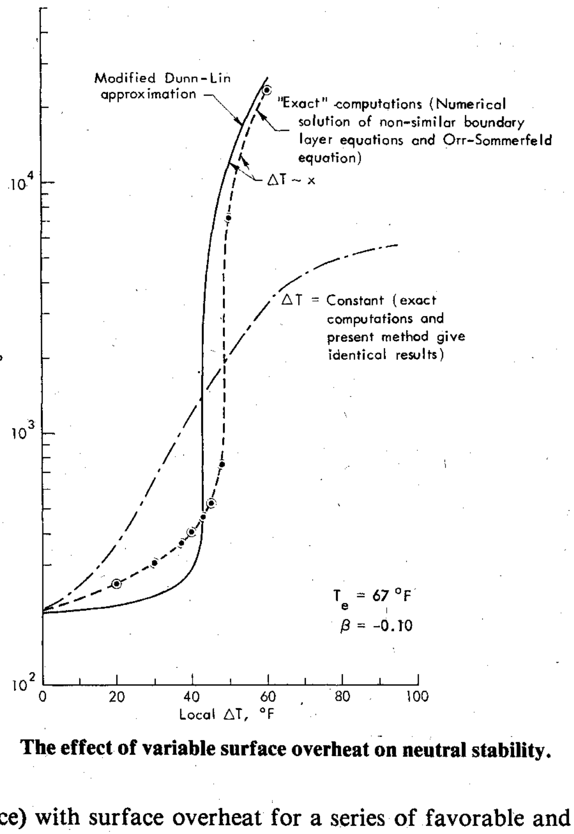

Fig. 3 The effect of variable surface overheat on neutral stability.

distance) with surface overheat for a series of favorable and unfavorable pressure gradients of the Falkner-Skan type. The symbols in this figure represent results obtained by numerical integration of the Orr-Sommerfeld equation, and the curves refer to the approximate method. The agreement between exact and approximate computations is again satisfactory in the practical range of β between $+0.10$ and -0.10. The results in Figs. 1 and 2 suggest that the method gives accurate results when applied to pressure gradients and surface overheats that are achievable in practice.

The results shown in Fig. 3 represent a severe challenge to the approximate method. Numerical solutions of the Orr-Sommerfeld equations for nonsimilar velocity profiles (the Falkner-Skan $\beta = -0.10$, but the wall temperature is linear in x) exhibit extreme sensitivity to surface overheat, particularly for the region $\Delta T > 4.44°C$ (40°F), where R_c^* increases sharply with ΔT. The approximate method not only duplicates this qualitative behavior, but considering its simplicity and economy, does it with remarkable accuracy. (More detailed analysis and further applications are described in Ref. 11; and Ref. 12 discusses unsteady flow stability.) The surprisingly powerful stabilizing synergism between wall temperature distribution and heating is due primarily to the ability of the thermal boundary layer to deform viscosity and velocity profiles in the region between critical layer and wall.

Acknowledgment

The research described in this report was sponsored by the Defense Advanced Research Projects Agency under Contract MDA903-78-C-0121.

References

[1]Wazzan, A. K. and Gazley, C., "The Combined Effects of Pressure Gradient and Heating on the Stability and Transition of Water Boundary Layers," *Proceedings of the 2nd International Conference on Drag Reduction*, London, 1977, BHRA Fluid Engineering, Cranfield, Bedford, England, 1977, p. E 3-23.

[2]Lin, C. C., "On the Stability of Two-Dimensional Parallel Flows," *Quarterly of Applied Mathematics*, Vol. III, No. 4, Pt. III, Jan. 1946, pp. 277-301.

[3]Lin, C. C., *The Theory of Hydrodynamic Stability*, Cambridge University Press, London, 1966.

[4]Lees, L. and Lin, C. C., "Investigation of the Stability of the Laminar Boundary Layer in a Compressible Fluid," NACA TN 1115, Sept. 1946.

[5]Dunn, D. W. and Lin, C. C., "On the Stability of the Laminar Boundary Layer in a Compressible Fluid," *Journal of the Aerospace Sciences*, Vol. 22, July 1955, pp. 455-477.

[6]Mack, L. M., "Numerical Calculation of the Stability of the Compressible Laminar Boundary Layer," Jet Propulsion Laboratory, Pasadena, Calif., Rept. 20-122, May 1, 1960.

[7]Wazzan, A. R., Okamura, T. T., and Smith, A.M.O., "The Stability and Transition of Heated and Cooled Incompressible Laminar Boundary Layers," *Proceedings of the International Heat Transfer Conference*, Paris-Versailles, Vol. II, Paper FC-1.4, 1970.

[8]Wazzan, A. R., Keltner, G., Okamura, T. T., and Smith, A.M.O., "Spatial Stability of Stagnation Water Boundary Layer with Heat Transfer," *Physics of Fluids*, Vol. 15, Dec. 1972, pp. 2114-2116.

[9]Wazzan, A. R., Okamura, T. T., and Smith, A.M.O., "Spatial and Temporal Stability Charts for the Falkner-Skan Boundary Layer Profiles," Douglas Aircraft Company, Long Beach, Calif., Rept. DAC-67086, Sept. 1968.

[10]Wazzan, A. R., Okamura, T. T., and Smith, A.M.O., "The Stability of Water Flow Over Heated and Cooled Flat Plates," *Journal of Heat Transfer*, Ser. C, Vol. 90, 1968, pp. 109-114.

[11]Aroesty, J., King, W. S., Harpole, G. M., Matyskiela, W. W., Wazzan, A. R., and Gazley, C. Jr., *Simple Relations for the Stability of Heated Laminar Boundary Layers in Water: Modified Dunn-Lin Method*, The Rand Corporation, Santa Monica, Calif., Rept. R-2209-ARPA, March 1978.

[12]King, W. S., Aroesty, J., Yao, L. S., and Matyskiela, W. W., "The Effects of Unsteady Potential Flow on Heated Laminar Boundary Layers in Water," *Proceedings of the 2nd International Conference on Drag Reduction*, London, 1977, BHRA Fluid Engineering, Cranfield, Bedford, England, 1977, p. E 4-41.